

On the exactness of the Lévy–transformation

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Abstract In a recent paper we gave a sufficient condition for the strong mixing property of the Lévy–transformation. In this note we show that it actually implies a much stronger property, namely exactness.

1 Introduction

Our aim in this short note, to supplement the result of [1]. In that work we obtained a condition which implies the strong mixing property, hence the ergodicity of the Lévy–transformation. We reformulate this condition, see (3) below, and show that it actually implies a stronger property called exactness. That is, we deduce that the tail σ -algebra of the Lévy transformation is trivial provided that condition (3) holds.

2 Summary of the results of [1]

First, we fix some notations. $\mathbf{W} = C[0, \infty)$ is the space of continuous function defined on $[0, \infty)$, \mathbf{P} is the Wiener measure on the Borel σ -field of \mathbf{W} , and β is the canonical process on \mathbf{W} . Finally T is a \mathbf{P} almost everywhere defined transformation of \mathbf{W} defined by the formula

$$(T\beta) = \int h(s, \beta) d\beta_s \quad (1)$$

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where h is a progressively measurable function on $[0, \infty) \times \mathbf{W}$ taking values in $\{-1, 1\}$. We use the notation $\beta^{(n)}$ for $T^n\beta$ and $(\mathcal{F}_t^{(n)})_{t \geq 0}$ for the filtration generated by $\beta^{(n)}$ and $h_s^{(n)} = \prod_{k=0}^{n-1} h(s, \beta^{(k)})$.

The transformation T is called *exact*, whenever $\bigcap_n \mathcal{F}_\infty^{(n)}$ is trivial.

The Lévy transformation is obtained by the choice $h(s, \beta) = \text{sign}(\beta_s)$ and denoted by \mathbf{T} . The rest of this section is devoted to this special case.

The main observation of [1] was that the existence of certain stopping times makes it possible to estimate the covariance of $h_s^{(n)}$ and $h_1^{(n)}$, which is the key to prove the strong mixing property of \mathbf{T} . More precisely, for $r \in (0, 1)$ and $C > 0$ let

$$\tau_{r,C} = \inf \left\{ s > r : \exists n, \beta_s^{(n)} = 0, \min_{0 \leq k < n} |\beta_s^{(k)}| > C\sqrt{(1-s)_+} \right\}.$$

That is $\tau_{r,C}$ is the first time after r when for some n the first n iterated paths are relatively far away from the origin while $\beta^{(n)}$ is zero.

Then it was proved that

$$\limsup_{n \rightarrow \infty} \left| \mathbf{E} \left(h_r^{(n)} h_1^{(n)} \right) \right| \leq \mathbf{P}(\tau_{r,C} = 1) + \mathbf{P} \left(\sup_{0 \leq s \leq 1} |\beta_s| > C \right). \quad (2)$$

It was stated without the first term on the right, under the assumption that this term is zero. The proof of this inequality used the coupling of the shadow path $\tilde{\beta}$, reflected after $\tau_{r,C}$ and the original path β . This argument actually yields the following form of (2)

$$\lim_{n \rightarrow \infty} \left| \mathbf{E} \left(h_1^{(n)} \mid \mathcal{F}_1^{(n)} \vee \mathcal{F}_r^{(0)} \right) \right| \leq \mathbf{P}(\tau_{r,C} = 1) + \mathbf{P} \left(\sup_{0 \leq s \leq 1} |\beta_s| > C \right). \quad (3)$$

Note that the limit on the left hand side exists as $\left| \mathbf{E} \left(h_1^{(n)} \mid \mathcal{F}_1^{(n)} \vee \mathcal{F}_r^{(0)} \right) \right|$ is a reversed submartingale.

By virtue of the estimates in (2) and (3) a sufficient condition for the strong mixing of the Lévy transformation is that

$$\tau_{r,C} < 1, \quad \text{almost surely, for all } r \in (0, 1), C > 0. \quad (4)$$

The main result of this paper is the following theorem.

Theorem 1. *If (4) holds then the Lévy transformation is exact.*

The proof is based on the estimate (3) and is given in the next section where we do not assume the special form of the Lévy transformation. That is, we prove the next statement from which Theorem 1 follows.

Proposition 2. *Let T be the transformation of the Wiener-space as in (1). If*

$$\lim_{n \rightarrow \infty} \mathbf{E} \left(h_t^{(n)} \mid \mathcal{F}_{rt}^{(0)} \vee \mathcal{F}_t^{(n)} \right) = 0, \quad \text{for almost all } t > 0 \text{ and } r \in [0, 1) \quad (5)$$

then T is exact.

3 Proof of Proposition 2.

For a deterministic function $f \in L^2([0, \infty))$ we will use the notation $\mathcal{E}(f)$ for

$$\mathcal{E}(f) = \exp \left\{ \int_0^\infty f(s) d\beta_s - \frac{1}{2} \int_0^\infty f^2(s) ds \right\}.$$

Since the linear hull of the set of $\{\mathcal{E}(f) : f \in L^2([0, \infty))\}$ is dense in $L^2(\mathbf{W})$ the following statement is obvious.

Proposition 3. *$\bigcap_n \mathcal{F}_\infty^{(n)}$ is trivial if and only if $\mathbf{E}(\mathcal{E}(f) \mid \mathcal{F}_\infty^{(n)}) \rightarrow 1$ for each $f \in L^2([0, \infty))$.*

To express $\mathbf{E}(\mathcal{E}(f) \mid \mathcal{F}_\infty^{(n)})$ we use the next proposition.

Lemma 4. *Assume that ξ is a measurable and $\mathcal{F}^{(0)}$ -adapted process satisfying $\mathbf{E}(\int_0^\infty \xi_s^2 ds) < \infty$. Then*

$$\mathbf{E} \left(\int_0^\infty \xi_s d\beta_s^{(0)} \mid \mathcal{F}_\infty^{(n)} \right) = \int_0^\infty \mathbf{E} \left(\xi_s h_s^{(n)} \mid \mathcal{F}_s^{(n)} \right) d\beta_s^{(n)}$$

Proof. First observe that both sides of the equation makes sense.

Denote by V the left hand side of the equation and by V' the right hand side. Besides let $U \in L^2(\mathcal{F}_\infty^{(n)})$ and write it, using that $\mathcal{F}^{(n)}$ is generated by the Brownian motion $\beta^{(n)}$, as $U = c + \int_0^\infty u_s d\beta_s^{(n)}$ with some $c \in \mathbb{R}$ and $\mathcal{F}^{(n)}$ -predictable u . Then

$$\begin{aligned} \mathbf{E}(UV) &= \mathbf{E} \left(\int_0^\infty \xi_s h_s^{(n)} u_s ds \right) \\ &= \mathbf{E} \left(\int_0^\infty \mathbf{E} \left(\xi_s h_s^{(n)} \mid \mathcal{F}_s^{(n)} \right) u_s ds \right) = \mathbf{E}(UV'). \end{aligned}$$

This proves that $V = V'$ which is the claim. \square

In the proof of the next statement we call a probability measure $Q \sim \mathbf{P}$ simple when it is in the form $dQ = \mathcal{E}(f)d\mathbf{P}$ with some $f \in L^2([0, \infty))$.

Proposition 5. $\bigcap_n \mathcal{F}_\infty^{(n)}$ is trivial if and only if for all $Q \sim \mathbf{P}$

$$\mathbf{E}_Q(h_s^{(n)} | \mathcal{F}_s^{(n)}) \rightarrow 0, \quad \mathbf{P}\text{-almost surely, for almost all } s > 0. \quad (6)$$

Proof. In the proof we mostly work with simple equivalent measures, and obtain the conclusion of the “only if” part by approximation.

First we get a formula for $\mathbf{E}(\mathcal{E}(f) | \mathcal{F}_\infty^{(n)})$ when $f \in L^2([0, \infty))$ and then we apply Proposition 3.

So for the simple equivalent measure $dQ = \mathcal{E}(f)d\mathbf{P}$, let the density process be denoted by $Z_t = \mathbf{E}(\mathcal{E}(f) | \mathcal{F}_t)$. Then $dZ_t = Z_t f(t) d\beta_t$ and by Lemma 4

$$\begin{aligned} Z_\infty^{(n)} &= \mathbf{E}(\mathcal{E}(f) | \mathcal{F}_\infty^{(n)}) = \mathbf{E}\left(1 + \int_0^\infty Z_t f(t) d\beta_t \middle| \mathcal{F}_\infty^{(n)}\right) \\ &= 1 + \int_0^\infty f(t) \mathbf{E}(Z_t h_t^{(n)} | \mathcal{F}_t^{(n)}) d\beta_t^{(n)}. \end{aligned}$$

By the Bayes rule $\mathbf{E}(Z_t h_t^{(n)} | \mathcal{F}_t^{(n)}) = \mathbf{E}_Q(h_t^{(n)} | \mathcal{F}_t^{(n)}) Z_t^{(n)}$. That is, with $\xi_t^{(n)} = \mathbf{E}_Q(h_t^{(n)} | \mathcal{F}_t^{(n)})$ and $M^{(n)} = \int \xi_s^{(n)} f(s) d\beta_s^{(n)}$ we can write

$$\mathbf{E}(\mathcal{E}(f) | \mathcal{F}_\infty^{(n)}) = \exp\left\{M_\infty^{(n)} - \frac{1}{2}\langle M^{(n)} \rangle_\infty\right\}.$$

When (6) holds then $\langle M^{(n)} \rangle_\infty \rightarrow 0$ in $L^1(\mathbf{P})$, $M_\infty^{(n)} \rightarrow 0$ in $L^2(\mathbf{P})$, hence $\ln Z_\infty^{(n)} \rightarrow 0$ in $L^1(\mathbf{P})$. Since $Z_\infty^{(n)} = \mathbf{E}(\mathcal{E}(f) | \mathcal{F}_\infty^{(n)})$ converges almost surely we get that its limit is 1. This is true for all $f \in L^2[0, \infty)$ and by Proposition 3 we obtain that the tail σ -field $\bigcap_n \mathcal{F}_\infty^{(n)}$ is trivial.

For the converse we prove below that when $\bigcap_n \mathcal{F}_\infty^{(n)}$ is trivial then for each $f \in L^2[0, \infty)$

$$f(s) \mathbf{E}(\mathcal{E}(f) h_s^{(n)} | \mathcal{F}_s^{(n)}) \rightarrow 0, \quad \text{almost surely, for almost all } s > 0. \quad (7)$$

Then we consider

$$\mathcal{H}_s = \left\{ \xi \in L^1(\mathbf{P}) : \mathbf{E}(\xi h_s^{(n)} | \mathcal{F}_s^{(n)}) \rightarrow 0 \text{ in } L^1(\mathbf{P}) \right\}, \quad s > 0.$$

\mathcal{H}_s is obviously a closed subspace of $L^1(\mathbf{P})$. It is possible to choose $D = \{f_1, f_2, \dots\} \subset L^2([0, \infty))$, a countable set of deterministic, nowhere vanishing functions, such that the linear hull of $\{\mathcal{E}(f) : f \in D\}$ is dense in $L^1(\mathbf{P})$. Finally let

$$\mathcal{T} = \left\{ s > 0 : \forall f \in D, \mathbf{E}(\mathcal{E}(f) h_s^{(n)} | \mathcal{F}_s^{(n)}) \rightarrow 0 \right\}.$$

Then \mathcal{T} has full Lebesgue measure within $[0, \infty)$ and for $s \in \mathcal{T}$ we obviously have $\mathcal{H}_s = L^1(\mathbf{P})$. For $s \in \mathcal{T}$ (6) follows, by considering $\xi = dQ/d\mathbf{P}$.

It remains to show that

$$\bigcap \mathcal{F}_\infty^{(n)} \text{ is trivial} \quad (8)$$

implies (7). So we fix f and use the notation $Q, \xi^{(n)}, M^{(n)}$ introduced at the beginning of the proof. Note that $(|\xi_s^{(n)}|, \mathcal{F}_s^{(n)})_{n \geq 0}$ is a reversed Q -submartingale for each fixed s . Hence $|\xi_s^{(n)}|$ is convergent almost surely (both under \mathbf{P} and Q by their equivalence) and the limit is $\bigcap_n \mathcal{F}_s^{(n)} \subset \bigcap_n \mathcal{F}_\infty^{(n)}$ measurable. Since $\bigcap_n \mathcal{F}_\infty^{(n)}$ is trivial there is a deterministic function g such that $|\xi_s^{(n)}| \rightarrow g(s)$ almost surely for almost all s . Obviously $0 \leq g(s) \leq 1$.

Another implication of (8) is that

$$\ln \mathbf{E}(\mathcal{E}(f) | \mathcal{F}_\infty^{(n)}) = M_\infty^{(n)} - \frac{1}{2} \langle M^{(n)} \rangle_\infty \rightarrow 0, \quad \text{almost surely.} \quad (9)$$

Here

$$\langle M^{(n)} \rangle_\infty \rightarrow \sigma^2 = \int_0^\infty (f(s)g(s))^2 ds, \quad \text{almost surely}$$

and we will see that $M_\infty^{(n)}$ has normal limit with expectation zero and variance σ^2 . Then (9) can only hold if $\sigma^2 = 0$ which obviously implies (7).

To finish the proof we write $M_\infty^{(n)}$ as

$$M_\infty^{(n)} = \int_0^\infty f(s)g(s) \text{sign}(\xi_s^{(n)}) d\beta_s^{(n)} + \int_0^\infty f(s)(\xi_s^{(n)} - g(s) \text{sign}(\xi_s^{(n)})) d\beta_s^{(n)}.$$

Here the law of the first term is normal $N(0, \sigma^2)$ not depending on n , while the second term goes to zero in $L^2(\mathbf{P})$. \square

To finish the proof of Proposition 2 assume that (5) holds, that is

$$\lim_{n \rightarrow \infty} \mathbf{E}(h_t^{(n)} | \mathcal{F}_{rt}^{(0)} \vee \mathcal{F}_t^{(n)}) = 0, \quad \text{for almost all } t > 0 \text{ and } r \in [0, 1].$$

Fix a $Q \sim \mathbf{P}$ and denote by $Z_t = \frac{dQ|_{\mathcal{F}_t^{(0)}}}{d\mathbf{P}|_{\mathcal{F}_t^{(0)}}}$ the density process. By the Bayes formula it is enough to show that

$$\mathbf{E}(Z_t h_t^{(n)} | \mathcal{F}_t^{(n)}) \rightarrow 0.$$

Since $|h^{(n)}| \leq 1$ we have the next estimate

$$\left\| \mathbf{E}(Z_t h_t^{(n)} | \mathcal{F}_{rt}^{(0)} \vee \mathcal{F}_t^{(n)}) - \mathbf{E}(Z_{rt} h_t^{(n)} | \mathcal{F}_{rt}^{(0)} \vee \mathcal{F}_t^{(n)}) \right\|_{L^1} \leq \|Z_t - Z_{rt}\|_{L^1},$$

and by (5)

$$\mathbf{E}\left(Z_{rt}h_t^{(n)} \mid \mathcal{F}_{rt}^{(0)} \vee \mathcal{F}_t^{(n)}\right) = Z_{rt}\mathbf{E}\left(h_t^{(n)} \mid \mathcal{F}_{rt}^{(0)} \vee \mathcal{F}_t^{(n)}\right) \rightarrow 0$$

almost surely and in L^1 . That is,

$$\limsup_{n \rightarrow \infty} \left\| \mathbf{E}\left(Z_t h_t^{(n)} \mid \mathcal{F}_t^{(n)}\right) \right\|_{L^1} \leq \inf_{r \in [0,1]} \|Z_t - Z_{rt}\|_{L^1} = 0.$$

This means that the limit of the reversed submartingale $|\mathbf{E}_Q(h_t^{(n)} \mid \mathcal{F}_t^{(n)})|$ is zero and T is exact by Proposition 5. This completes the proof of Proposition 2.

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References

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